A NOTE ON SPHERICAL SUMMATION MULTIPLIERS

BY

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ABSTRACT

We give a new proof of a theorem of L. Carleson and P. Sjölin on L^p-boundedness of spherical summation operators in two variables.

In this article we present alternate proofs of two recent multiplier theorems for Bochner-Riesz spherical summation operators. These operators are defined on $L^p(R^n)$ by the equations $\widehat{T_\lambda f}(\xi) = m_\lambda(\xi)\hat{f}(\xi)$, where $m_\lambda(\xi) = (1 - |\xi|^2)^\lambda$ if $|\xi| < 1$, and $m_\lambda(\xi) = 0$ if $|\xi| \geqq 1$. In the limiting case $\lambda = 0$, we take m_λ to be the characteristic function of the unit ball. T_0 is a natural n-variable analogue of the Hilbert transform, while for $\lambda > 0$, T_λ is a basic operator that serves to define Cesaro summation of multiple Fourier integrals. The multiplier problem for T_λ simply asks: For which λ and p is T_λ a bounded operator on $L^p(R^n)$?

A great deal is already known about the multiplier problem for T_λ (see, e.g., Bochner [1], Stein [8], Herz [5], Fefferman [3], [4], and Carleson and Sjölin [2]), and the known results are summarized below.

THEOREM.

(A) (Herz [5]). *T_λ is unbounded on $L^p(R^n)$ unless p lies strictly between $p_0(\lambda) = 2n/(n + 1 + 2\lambda)$ and its dual exponent $p_0'(\lambda)$.*

(B) (Fefferman [3]). *If $\lambda > (n - 1)/4$, then T_λ is bounded on $L^p(R^n)$ for $p_0(\lambda) < p < p_0'(\lambda)$.*

(C) (Carleson and Sjölin [2]; see also a simplified version by Hörmander [6]). *In R^2, T_λ is bounded on L^p whenever $\lambda > 0$ and $p_0(\lambda) < p < p_0'(\lambda)$.*

(D) (Sjölin [7]). *In R^3, T_λ is bounded on L^p whenever $\lambda > \frac{1}{4}$ and $p_0(\lambda) < p < p_0'(\lambda)$.*

(E) (Fefferman [4]). *T_0 is never bounded on $L^p(R^n)$ unless $n = 1$ or $p = 2$.*

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Thus, although new ideas are needed for a complete solution, it seems likely that the answer to our multiplier problem will be that T_λ is bounded on L^p if and only if $\lambda > 0$ and $p_0(\lambda) < p < p_0'(\lambda)$.

Unfortunately, the proofs of (A)–(E) now in the literature have so little in common with one another, that one suspects that we do not yet understand what is really going on. (This is especially striking for the two sharp results, (C) and (E); (C) was proved in the spirit of the Hausdorff-Young inequality, while (E) is based on the Kakeya needle problem.) Actually, things are not so bleak (we hope). The main object of this article is to present the author's proof of (C), which has points of close contact with all the various known proofs of (A)–(E), and perhaps helps to show how they fit into a coherent whole. We also take the opportunity to present E. M. Stein's elegant and simple proof of (B) and (D).

Before we come to the details of the proofs, however, some remarks are in order concerning the underlying strategy. We may single out three main ideas which lurk behind the proofs of (A)–(E).

(1) The multiplier problem for T_λ is intimately related to some surprising "restriction theorems" on Fourier transforms. A typical restriction theorem is as follows: Let f be a function in $L^{4/3-\varepsilon}(R^2)$. Then its Fourier transform $\hat{f}$ belongs to $L^{4/3}$ when restricted to the unit circle. (For a precise statement and the easy proof, see [3]). This result makes essential use of the curvature of the circle, and the analogous statement for a line segment is completely false. Restriction theorems serve as a simplified model for the phenomena that arise in the study of T_λ. To illustrate the close connection between restrictions and T_λ, we mention an easy folk theorem, which the reader may verify with a little calculation: Let $f \in L^p(R^n)$ have compact support. Then

$$T_0 f(x) \sim \frac{e^{i|x|}}{|x|^{(n+1)/2}} \cdot \hat{f}\left(\frac{x}{|x|}\right) \text{ as } x \to \infty.$$

Actually, this observation is only a beginning, and the more one thinks about T_λ, the more one is led inexorably to the analogy between multiplier theorems and restriction theorems.

(2) In two dimensions, it happens that the crucial inequality for T_λ is an L^4-estimate. (See below.) Now, to decide whether a function F belongs to L^4, we may simply ask whether F^2 is in L^2. This allows us to use L^2-methods (orthogonality, the Fourier transform, etc.) to reduce our basic L^4-estimate to something

easy, which is the point of proving estimate (6) below. Already in R^3, the crucial problem is an L^3-estimate, and we can no longer reduce matters to L^2. Thus, new ideas are needed to make progress in higher dimensions.

(3) In the standard one-dimensional case, one knows that the Hilbert transform is linked inseparably to the Hardy-Littlewood maximal function. Thus, in trying to arrive at a real understanding of the operators T_λ, one ought to look for an n-dimensional variant of the max function and study its connection with T_λ. Now it appears that the correct maximal function with which to compare T_λ is not the usual one formed with cubes or balls, but rather some sort of "Kakeya" maximal function, formed from highly eccentric rectangles pointing in arbitrary directions. We have only begun to explore this approach, but so far it has yielded interesting results (see [4]); perhaps more will come of it. The real purpose of our proof of (C) is to begin the task of seeing how T_λ may be controlled by a Kakeya maximal function.

We are now ready to prove the result of Carleson and Sjölin.

THEOREM 1. *All* T_λ $(\lambda > 0)$ *are bounded on* $L^4(R^2)$.

From this, the full result (C) follows easily by interpolation and duality.

PROOF. T_λ is a convolution operator with a kernel k_λ which is essentially $k_\lambda(x) = e^{i|x|}/|x|^{3/2+\lambda}$ see [5]. Immediately we decompose k_λ into $K^0(x) + \sum_{l=1}^{\infty} K^l(x)$, where $K^l(x) = \phi^l(|x|) \cdot k_\lambda(x)$, and $\{\phi^l\}$ is a smooth partition of unity on $(0,\infty)$ with ϕ^1 supported in $(1,3)$ and $\phi^l(r) = \phi^1(2^{1-l}r)$ for $l \geqq 1$. K^0 is an L^1 kernel, so we neglect it. We shall prove that the operator norm of $T^l: f \to K^l * f$ on L^4 is at most

$$(1) \qquad \|T^l\| \leqq C \cdot 2^{-l\eta} \qquad (\text{some } \eta > 0).$$

Once we know this, we obtain the Carleson-Sjölin result at once, simply by summing a geometric series.

Let us fix l, and set $N = 2^{l+2}$. To prove the estimate (1), we shall make several successive decompositions of K^l into ever-finer pieces. But for all our merry pranks, we shall never decompose K^l into more than N^3 pieces. Therefore, as we go about estimating the various pieces of K^l, no error terms whose norms are $O(N^{-500})$ will make significant contributions, and we refer to such miniscule terms as "negligible". Thus, we systematically ignore negligible terms.

Now let us start to prove (1). Divide the plane R^2 into a grid of N by N squares. Since the support of the kernel K^l has diameter $\leqq N$, it follows that T^l in effect acts independently on each of the squares of the grid, so to prove estimate (1), it is enough to consider functions f defined on a single N by N square Q^0 and prove that

(2) $$\|K^l * f\|_{L^4(Q^0)} \leqq CN^{-\eta}\|f\|_{L^4(Q^0)}.$$

We now make our first decomposition of K^l. Let $\{\beta_j(\theta)\}_{|j|\leqq \pi N^{\frac{1}{2}}}$ be a smooth partition of unity, $1 = \sum_j \beta_j(\theta)$ in $|\theta| \leqq \pi$, with β_j supported in $(j-1)/N^{1/2} \leqq \theta \leqq (j+1)/N^{1/2}$. Then using polar coordinates in the plane, decompose K^l into $K^l(r) = \sum_{|j|\leqq \pi N^{\frac{1}{2}}} K^l(r)\beta_j(\theta) \equiv \sum_j K^l_j(r.\theta)$. To simplify the Euclidean geometry that will come into our proof later, we shall replace K^l and T^l by

$$\tilde{K}^l = \sum_{0\leqq j\leqq (\pi/4)N^{\frac{1}{2}}} K^l_j$$

and its corresponding convolution operator $\tilde{T}^l$. In effect, $\tilde{K}^l(r,\theta) = \chi_{[0,\pi/4]}(\theta)K^l(r)$. Instead of (2), we shall prove the analogous estimate

(3) $$\|\tilde{T}^l f\|_{L^4(Q^0)} \leqq CN^{-\eta}\|f\|_{L^4(Q^0)}.$$

From (3), estimate (2) follows at once, since the kernel K^l is a sum of eight isomorphic terms $K^l(r) = \chi_{[0,\pi/4]}(\theta)K^l(r) + \chi_{[\pi/4,\pi/2]}(\theta)K^l(r) + \cdots + \chi_{[7\pi/4,2\pi]}(\theta)K^l(r)$.

Next we introduce some notation. Each $K^l_j(r,\theta)$ is supported in a "rectangle" $R_j = \{(r,\theta)\,|\,N/4 \leqq r \leqq N,\ (j-1)/N^{1/2} \leqq \theta \leqq (j+1)/N^{1/2}\}$. Let ω_j be the unit vector in the direction $\theta = j/N^{1/2}$, and let ν_j be a unit vector perpendicular to ω_j. The "rectangle" R_j then has dimension roughly $N^{1/2}$ in the ν_j-direction and N in the ω_j-direction.

To prove (3), we need information on the Fourier transform of K^l_j, and so we define some auxiliary functions on R^2. Take a small number $\delta > 0$ to be determined later, and define smooth functions ψ_j and ϕ_j so that

(α) $\phi_j(x) = 1$ for x in the rectangle $$\begin{cases} |(x-\omega_j)\cdot\omega_j| \leqq N^{\delta-1} \\ |x\cdot\nu_j| \leqq N^{\delta-1/2}, \end{cases}$$

and $\phi_j(x) = 0$ outside the double of that rectangle,

(β) $\psi_j(x) = 1$ if $|x-\omega_j| \leqq 3N^{\delta-1/2}$, but $\psi_j(x) = 0$ if $|x-\omega_j| \geqq 5N^{\delta-1/2}$.

In effect, we shall see that $\hat{K}^l_j$ is supported inside $\operatorname{supp}(\phi_j)$. More precisely, $K^l_j = K^l_j * \hat{\phi}_j +$ a negligible term. To see this, we start with the function $e^{-ix\cdot\omega_j}K^l_j(x)$, and dilate the plane by a factor of N^{-1} in the ω_j-direction and $N^{-1/2}$ in the ν_j-direction. We arrive at a function k_{lj} in the Schwartz class (R^2), with bounds independent of l and j. Consequently, if N is large, the Fourier transform $\hat{k}_{lj}$ will be supported in $|\xi| \leqq N^{\delta}$, except for a negligible error. Passing from k_{lj} back to K^l_j, we find exactly that $K^l_j = K^l_j * \hat{\phi}_j +$ negligible.

Now we can begin to prove (3). Modulo negligible terms, we have

$$\| \tilde{T}^l f\|_4^4 = \| \sum_j K^l_j * \hat{\phi}_j * f\|_4^4 = \| \sum_{j,j'} (K^l_j * \hat{\phi}_j * f) \cdot (K^l_{j'} * \hat{\phi}_{j'} * f) \|_2^2$$
$$= \| \sum_{j,j'} (\hat{K}^l_j \phi_j \hat{f}) * (\hat{K}^l_{j'} \phi_{j'} \hat{f}) \|_2^2$$

by the Plancherel theorem. Now the support of $(\hat{K}^l_j \phi_j \hat{f}) * (\hat{K}^l_{j'} \phi_{j'} \hat{f})$ is contained in $A_{jj'} =$ support $(\phi_j) +$ support$(\phi_{j'})$, and by the geometry of the situation, no point of R^2 belongs to more than $N^{3\delta}$ of the $A_{jj'}$. (This is where we use the fact that the circle is curved; cf. the restriction theorem in [3]). It follows that

$$\| \tilde{T}^l f\|_4^4 \leqq N^{3\delta} \sum_{j,j'} \| (\hat{K}^l_j \phi_j \hat{f}) * (\hat{K}^l_{j'} \phi_{j'} \hat{f}) \|_2^2$$
$$= N^{3\delta} \sum_{j,j'} \| (K^l_j * \hat{\phi}_j * f) \cdot (K^l_{j'} * \hat{\phi}_{j'} * f) \|_2^2 \text{ (by Plancherel again)}$$
$$= N^{3\delta} \sum_{j,j'} \|(K^l_j * \hat{\phi}_j * \hat{\psi}_j * f) \cdot (K^l_j * \hat{\phi}_{j'} * \hat{\psi}_{j'} * f) \|_2^2, \quad \text{since} \quad \hat{\phi}_j = \hat{\phi}_j * \hat{\psi}_j,$$

and modulo negligible terms this gives

$$\| \tilde{T}^l f\|_4^4 \leqq N^{3\delta} \sum \|(K^l_j * \hat{\psi}_j * f) \cdot (K^l_{j'} * \hat{\psi}_{j'} * f) \|_2^2. \tag{4}$$

This completes the first step in our proof of (3). Having cut up K^l, we are now going to decompose f. First we divide our N by N square into $N^{1/2}$ vertical strips $P^1, P^2, \cdots, P^{N^{\frac{1}{2}}}$, of dimensions $N^{1/2}$ by N. We shall prove that if f is a function supported on a single strip P^s, then

$$\| \tilde{T}^l f\|_4 \leqq CN^{-3/4-\eta} \|f\|_4. \tag{5}$$

This estimate yields (3) at once, for we simply write $f \in L^4(Q^0)$ as $f = \sum_{s \leqq N^{\frac{1}{2}}} f_s$ with f_s supported on P^s, and then from (5), we have

$$\| \tilde{T}^l f\|_4 \leqq \sum_{s \leqq N^{\frac{1}{2}}} \| \tilde{T}^l f_s\|_4 \leqq CN^{-3/8-\eta} \sum_{s \leqq N^{\frac{1}{2}}} \|f_s\|_4$$
$$\leqq CN^{-3/8-\eta} \cdot (N^{1/2})^{3/4} (\sum_s \|f_s\|_4^4)^{1/4} \leqq CN^{-\eta} \|f\|_4.$$

So we shall fix a strip P^0 once and for all, and try to prove (5) for functions supported on P^0.

Next, we cut up the strip P^0 into $N^{1/2}$ small squares $\{Q_i\}$ of size $N^{1/2}$ by $N^{1/2}$, and split $f \in L^4(P^0)$ into $f = \sum f_i$ with f_i supported in Q_i. Set $f_{ij} = \hat{\psi}_j * f_i$. Modulo a negligible error, f_{ij} lives in Q_i^*, the double of Q_i. Now by (4) we have

$$\| \tilde{T}^l f\|_4^4 \leqq N^{3\delta} \sum_{j,j'} (\| K_j^l * \hat{\psi}_j * f) \cdot (K_{j'}^l * \hat{\psi}_{j'} * f) \|_2^2$$
$$= N^{3\delta} \sum_{j,j'} \| \sum_{i,i'} (K_j^l * f_{ij}) \cdot (K_{j'}^l * f_{i'j'}) \|_2^2.$$

For each fixed j,j', the i,i'-term in the inner sum is supported (modulo neligible errors) in the set

$$P_{ii'jj'} = [\text{support}(K_j^l) + Q_i^*] \cap [\text{support}(K_{j'}^{l'}) + Q_{i'}^*].$$

Moreover, it is a simple geometrical fact that for fixed j,j', no point of R^2 belongs to more than ten of the sets $P_{ii'jj'}$. Therefore,

$$\| \tilde{T}^l f\|_4^4 \leqq CN^{3\delta} \sum_{j,j'} \sum_{i,i'} \| (K_j^l * f_{ij}) \cdot (K_{j'}^l * f_{i'j'}) \|_2^2. \tag{6}$$

This is our basic estimate; from here on, our proof of (5) is rather straightforward.

Let us make crude estimates for f_{ij} and $K_j^l * f_{ij}$. First of all, the Plancherel formula and Hölder's inequality show at once that

$$\sum_j \|f_{ij}\|_2^2 = \sum_j \|\hat{\psi}_j * f_i\|_2^2 = \int_{R^2} \sum_j \psi_j^2 \; |f_i|^2 d\xi \leqq N^\delta \int_{R^2} |f_i|^2 d\xi$$
$$= N^\delta \|f_i\|_2^2 \leqq N^{\delta + 1/2} \|f_i\|_4^2.$$

In particular, for $c_{ij} = (1/|Q_i|) \int_{Q^*} | f_{ij}(y)| \, dy$, we have the estimates $\sum_j |c_{ij}|^2 \leqq CN^{\delta - 1/2} \|f_i\|_4^2$ (by Minkowski's inequality); hence,

$$\sum_j (\sum_j c_{ij}^2)^2 \leqq CN^{2\delta - 1} \sum_i \|f_i\|_4^4 = CN^{2\delta - 1} \|f\|_4^4. \tag{7}$$

On the other hand, we can estimate $K_j^l * f_{ij}$ in terms of c_{ij} by writing $| K_j^l * f_{ij}(x)| \leqq \int_{Rj} \| K_j^l\|_\infty |f_{ij}(x-y)| \, dy \leqq N^{-1/2-\lambda} \; c_{ij} \chi_{Qi^* + Rj}(x)$, since $R_j = \text{supp } (K_j^l)$. Putting this estimate into (6) yields

$$\| \tilde{T}^l f\|_4^4 \leqq CN^{3\delta - 1 - 4\lambda} \sum_{j,j'} \sum_{i i'} c_{ij}^2 c_{i'j'}^2 \cdot \text{area}([Q_i^* + R_j] \cap [Q_{i'}^* + R_{j'}])$$
$$\equiv CN^{3\delta - 1 - 4\lambda} \sum_{ii'jj'} c_{ij}^2 c_{i'j'}^2 \cdot \text{area}(P_{ii'jj'}).$$

An elementary calculation of the area of $P_{ii'jj'}$ shows that:

(a) $P_{ii'jj'}$ is empty unless $\frac{1}{2} \leqq |i - i'|/|j - j'| \leqq 2$.

(b) If $P_{ii'jj'}$ is nonempty, its area is at most $CN^{3/2}/(|i - i'| + 1)$. (Here we assume the Q_i's are numbered consecutively, top to bottom.) Consequently,

(8) $$\| \tilde{T}^l f \|_4^4 \leq C N^{3\delta - 1/2 - 4\lambda} \sum_{i,i',j,j'=1}^{N^{1/2}} \frac{c_{ij}^2 c_{i'j'}^2}{|i - i'| + 1},$$

and we already have an estimate (7) for the c_{ij}. The right-hand side of (8) may be rewritten in the form

$$C N^{3\delta - 1/2 - 4\lambda} \sum_{i,i'=1}^{N^{1/2}} \frac{\left(\sum_j c_{ij}^2 \right) \left(\sum_j c_{i'j'}^2 \right)}{|i - i'| + 1},$$

and we now invoke the elementary inequality

$$\sum_{i,i'=1} \frac{A_i A_{i'}}{|i - i'| + 1} \leqq C \log N \cdot \sum_i A_i^2,$$

with $A_i = \sum_j c_{ij}^2$. The result is

$$\| \tilde{T}^l f \|_4^4 \leqq C N^{3\delta - 1/2 - 4\lambda} \log N \cdot \sum_i \left(\sum_j c_{ij}^2 \right)^2 ,$$

and at last by (7) we have

(9) $$\| T^l f \|_4^4 \leqq C N^{3\delta - 3/2 - 4\lambda} \log N \cdot \| f \|_4^4 .$$

This implies (5) at once, since the small number δ is at our disposal; and since we had reduced matters to (5), the proof of the Carleson-Sjölin theorem is complete. Q.E.D.

REMARK. By keeping careful track of error terms, we could in effect have taken $\delta = 0$ in the above. Equation (9) would then show that, as an operator on $L^4(R^2)$, T_λ^l has norm $\| T_\lambda^l \| = O(N^{-\lambda} (\log N)^{1/4})$. This bound is rather sharp; in particular, the techniques of [4] show that

$$\| T_\lambda^l \| \geqq \frac{c N^{-\lambda} (\log N)^{1/4}}{\log \log N} \text{ as } l \to \infty. \qquad \text{(Recall that } N = 2^l\text{)}.$$

We conclude with E. M. Stein's simple proof of (B) and (D). The original proof of (B) given in [3] involved a rather complicated reduction to the following L^2 restriction theorem: If $f \in L^p(R^n)$ and $p > 4n/(3n+1)$, it follows that $\hat{f}$ belongs to L^2 on the unit sphere, and

(10) $$\| \hat{f} \|_{L^2(S^{n-1})} \leqq C \| f \|_p .$$

To prove (D), Sjölin replaced (10) with a sharper estimate, valid in R^3, namely,

$$\|\hat{f}\|_{L^2(S^2)} \leqq C\|f\|_{4/3}. \tag{11}$$

Stein's proof of (B) and (D) consists of the following.

LEMMA. *Suppose that $\lambda > 0$ and $p_0(\lambda) < p < 2$. Suppose also that inequality* (10) *holds for all $f \in L^p(R^n)$. Then T_λ is bounded on $L^p(R^n)$.*

PROOF. As in the proof of Theorem 1, we split T_λ into $T_\lambda^0 + \sum_{l=1}^{\infty} T_\lambda^l$ with K_λ^l the convolution kernel for T_λ^l, and we note that the lemma reduces to the estimate

$$\| T_\lambda^l f\|_{L^p(Q^0)} \leqq C \cdot 2^{-l\eta}\|f\|_p \quad \text{(for some } \eta > 0\text{)}, \tag{12}$$

for f supported in a cube Q^0 of side $\sim 2^{l+2}$.

We prepare to apply the L^2 restriction theorem (10). First, observe that by a simple change of variable, we may replace the unit sphere in (10) by the sphere of radius r, and (10) will hold uniformly for $\frac{1}{2} \leqq r \leqq 2$. Secondly, some calculations with Hankel transforms show that $\hat{K}_\lambda^l(\xi)$ is a radial function supported in $\frac{1}{2} \leqq |\xi| \leqq 2$, except for a negligible error.

Now to prove (12) we simply write

$$\begin{aligned}
\| T_\lambda^l f\|_{L^p(Q^0)}^2 &\leqq 2^{nl[2/p-1]}\| T_\lambda^l f\|_2^2 = 2^{nl[2/p-1]}\int_{R^n} |\hat{K}_\lambda^l(\xi)|^2 |\hat{f}(\xi)|^2 d\xi \\
&= 2^{nl[2/p-1]}\int_0^\infty r^{n-1}|\hat{K}_\lambda^l(r)|^2 \left(\int_{\omega \in S^{n-1}} |\hat{f}(r\omega)|^2 d\omega\right) dr \\
&\leqq C2^{nl[2/p-1]}\|f\|_p^2 \int_0^\infty r^{n-1}|\hat{K}_\lambda^l(r)|^2 dr = C2^{nl[2/p-1]}\|f\|_p^2 \| K_\lambda^l\|_2^2 \\
&\leqq C2^{l\{n(2/p-1)-(1+2\lambda)\}}\|f\|_p^2,
\end{aligned}$$

if we recall the size of the kernel K_λ^l. If $p > p_0(\lambda)$, the expression in braces is negative. Q.E.D.

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